Int. J. Solids Structures Vol. 16, pp. 807–823
© Pergamon Press Ltd., 1980. Printed in Great Britain

0020-7683/80/0901–0807/$02.00/0

A THEORY FOR STABLE CRACK EXTENSION RATES IN DUCTILE MATERIALS

EDWARD W. HART
Cornell University, Ithaca, NY 14853, U.S.A.

(Received 11 July 1979; in revised form 7 January 1980)

Abstract—A novel theory is presented for the growth of cracks in ductile materials. The theory predicts the steady state crack tip extension velocity as it depends on the remote loading, geometry, and material parameters. The theory is based on (a) a kinetic law for the crack tip velocity v as it depends on the stress state at the crack tip and (b) a realistic real time flow rate law for the material non-elastic deformation.

It is shown that with a rate flow law and for a finite crack tip velocity, no matter how small, the crack tip stress field is characterized by a $r^{-1/2}$ singularity. This is contrary to the case for a strictly stationary crack. Because of this stress behavior, the Irwin crack extension force is again available as a driving force in the crack tip kinetic law noted above.

The theory predicts a broad range of crack extension phenomena such as are commonly observed. Steady state threshhold values and "critical" values for apparent stress intensity factors arise in the theory from stability limits of the predicted steady state curves.

1. INTRODUCTION

The subject of this paper is an entirely novel theory for the growth of cracks in ductile materials. Specifically, it is a theory for the steady state crack tip extension velocity v as it depends on the remote loading of the body, the crack geometry, and certain material properties of the body.

The principal novel feature of the theory is the use of the mechanical analysis of crack equilibrium due to Griffith [1] and Irwin [2]. In particular the Irwin *crack extension force* $\mathcal{G}$ plays a central role comparable to its significance in brittle fracture. We shall show that the Griffith–Irwin theory can be consistently extended into the non-equilibrium regime if the crack extension process and the plastic flow process are both treated as *real time rate processes*. We note that we are not concerned here with dynamical or inertial effects at all but rather with kinetic rate laws and dissipative phenomena.

The importance of the Irwin analysis [2] was that he showed that a reversible crack extension force was determined entirely by the local stress state at the crack tip. He characterized that stress state by the *stress intensity factor* K that measured the strength of the $r^{-1/2}$ stress singularity at the crack tip. It was shown by Barenblatt [3] that even in an atomistic cohesive model the continuum singularity characterization was still applicable to determine the force $\mathcal{G}$. The importance of this is the *local* nature of the crack extension force. In the present paper we shall show how the *local stress intensity factor* K can be distinguished from the *apparent stress intensity factor* K_A that would be deduced from the remote loading and the crack geometry under the assumption of linear elastic behavior alone. The distinction will arise from the stress changes in the cracked body due to the plastic deformation of the actual material of the body.

At this point it might be questioned whether, for ductile materials, the $r^{-1/2}$ singularity characteristic of the elastic stress field is even present. It is well known that for power law hardening materials (as well as for power law non-Newtonian viscosity) the index of the singularity changes, as shown by Hutchinson [4] and Rice and Rosengren [5]. The calculation of the present paper will show that, for realistic, rate-dependent flow laws, the $r^{-1/2}$ stress singularity is preserved for crack tips that are moving at some velocity v no matter how small v is. The solution for the moving crack is, in fact, non-analytic in v at $v = 0$. Thus the static crack problem is actually a singular problem.

For simplicity, the theory will be presented for a semi-infinite crack configuration and for the case of anti-plane strain known as Mode III. In this paper we shall omit the customary subscript III on the symbols $\mathcal{G}$ and K.

Although we present the theory for the case of the sharp crack in Mode III, it is expected that the principal conclusions will be applicable also for more complex structures.

2. PHYSICAL BACKGROUND FOR THE THEORY

In addition to the laws of mechanics there are two essential ingredients of the theory that relate to assumed or measurable material constitutive properties. These are (a) a kinetic law for the crack extension rate v as a function of the local value of $\mathcal{G}$ and (b) a constitutive law for the *non-elastic strain rate* $\dot{\epsilon}$ as a function of the local stress deviator σ. Since neither of these has wide currency in the form we shall employ, we shall now discuss them.

(a) *Crack propagation kinetics*

The theory of crack equilibrium of Griffith[1] for brittle solids was modified by Irwin[2] in a way that made explicit the local nature of the equilibrium at the crack tip. Irwin showed that a generalized force, the crack extension force $\mathcal{G}$, could be defined in terms of the *local stress state* at the crack tip and that the equilibrium could be expressed as the equality between $\mathcal{G}$ and the *reversible work* of crack extension per unit distance $\mathcal{G}_0$ that can be expressed in terms of the free energy of the new surface generated by cracking. Thus, the crack is in equilibrium with the applied forces when

$$\mathcal{G} = \mathcal{G}_0. \tag{1}$$

The Griffith–Irwin theory is a theory of equilibrium for quasi-static crack tip displacements and it says nothing about crack propagation at finite velocities.

The problem of finite velocities is completely analogous to that of other kinetic phenomena that are common in materials science. In our case, any finite value of v results in some rate of energy dissipation over the reversible work rate and so $\mathcal{G}$ must exceed $\mathcal{G}_0$ by such an amount that the mean dissipation rate $\mathcal{D}$ satisfies the relation

$$\mathcal{D} = (\mathcal{G} - \mathcal{G}_0)v. \tag{2}$$

This dissipation rate should not be confused with any dissipation due to non-elastic deformation. The cracking process and the plastic flow process are not statistically correlated phenomena. They interact only through the field equations that govern the stress equilibrium.

The dissipation with which we are concerned here arises from the atomic energy barriers between successive local equilibrium, positions of the crack. Such barriers to propagation of defect configurations are common in problems involving the lattic mechanics of singular lattice structures and reflect the translation periodicity of the lattice. The crack motion problem is entirely analogous to the well known problem for crystal dislocations as first described by Peierls[6]. The phenomenon for crystal cracks was first discussed explicitly by Thomson *et al.*[7]. It was in fact already implicit in the Barenblatt theory[3], further discussed by Barenblatt *et al.*[8], and also in the work of Gehlen and Kanninen[9] on computer simulation of crystal cracks. The first attempt at deduction of a kinetic law for v is that of Hsieh and Thomson[10].

The subject has been treated in detail by Lawn and Wilshaw[11] in a book that details the kinetic problem. Lawn[12] has further described a derivation of a kinetic law for the crack tip velocity analogous to that which has been deduced for crystal dislocations.

The main purport of this current work is that there must exist a kinetic law for the velocity of a crack tip as it depends on $\mathcal{G}$, $\mathcal{G}_0$ and the temperature T, based on thermal activation. Thus we may write generally

$$v = v(\mathcal{G}, \mathcal{G}_0; T). \tag{3}$$

We emphasize again that, for the theory presented here, the $\mathcal{G}$ that appears in eqn (3) is that which is determined by the stress intensity factor K that characterizes the local stress singularity at the crack tip regardless of the nature of the stress field elsewhere in the body. If the body is strictly elastic, that K will be the same as K_A determined from the applied loads, but, if non-elastic deformation has resulted in self-stress in the body, K is not the same as K_A.

We shall choose an explicit form for the kinetic function of eqn (3) below for the purpose of explicit calculation.

(b) *Constitutive law for non-elastic flow*

We stated above that we were concerned with "realistic, rate-dependent flows laws". What is meant by that characterization is constitutive relations for non-elastic deformation that more or less accurately reflect the behavior of metals according to modern testing experience. The principal features of modern constitutive relations are that the non-elastic strain rate is uniquely determined at any time by the current value of the stress deviator $\boldsymbol{\sigma}$ and by the current values of one or more internal state variables that account for the prior deformation history through incremental evolution laws. An example of such a theory is that due to Hart[13].

We shall in the present paper avoid complications that might arise from using constitutive relations that are too complex. Fortunately, the principal form of the solution of our problem seems to depend only slightly on the details of the constitutive relations. We shall, therefore, require only the simplest general structure for them.

We require specifically that the non-elastic strain rate $\dot{\boldsymbol{\epsilon}}$ and the stress deviator $\boldsymbol{\sigma}$ be related through a simple isotropic relation of the form

$$\dot{\boldsymbol{\epsilon}} = (\dot{\epsilon}/\sigma)\boldsymbol{\sigma}, \tag{4}$$

where the light faced symbols stand for the simple second invariants of their respective tensors of the form

$$\dot{\epsilon} \equiv +\sqrt{(\dot{\epsilon}_{ij}\dot{\epsilon}_{ij})}, \tag{5}$$

and where the summation convention is implied for repeated indices. A similar equation holds for σ.

Now the remainder of the constitutive equations consists of scalar relations

$$\dot{\epsilon} = \dot{\epsilon}(\sigma, \sigma^*), \tag{6}$$

where σ^* is a single hardness state variable satisfying the evolution law

$$\dot{\sigma}^* = \dot{\epsilon}F(\sigma, \sigma^*). \tag{7}$$

The functions $\dot{\epsilon}(\sigma, \sigma^*)$ and $F(\sigma, \sigma^*)$ must be specified. The relationships, eqns (4)–(7), are rather a simplification from the complete relations described in Ref. [13]. They nevertheless are fairly effective, especially when specialized to use for moderate and low temperature behavior for metals. In our present treatment we shall actually ignore the strain hardening given by eqn (7) and restrict our attention to the strain rate behavior at constant σ^*. This makes our material effectively a non-Newtonian viscous body. The strain hardening, which really modifies the results rather slightly, will be the subject for a later paper.

We propose then to use in our calculation, any of many possible forms for the function $\dot{\epsilon}(\sigma, \sigma^*)$. We shall see that the calculation is rather model independent and the only effect of specific models is numerical. We shall further restrict our constitutive function $\dot{\epsilon}$ by the moderate requirement that it have the form

$$\dot{\epsilon} = \dot{\epsilon}^*(\sigma^*)\phi(\sigma/\sigma^*). \tag{8}$$

So long as σ^* is constant, this is actually no restriction.

3. MECHANICAL BASIS FOR THE THEORY

We shall calculate, in the next section, a boundary value problem involving non-linear non-elastic flow. Because of the determinative form of the flow law, it is convenient to formulate and solve this problem by use of the continuum dislocation theory of self-stress. Such a method for solving plasticity problems was first proposed by Mura[14], but there does not seem to have been any subsequent application of the method. We shall, therefore, briefly describe the basis for this method of solution in this section.

The method is based on the theory for incompatible non-elastic strain. The relationship between incompatibility and continuum dislocation was given a definitive form by Kröner[15].

Because of the non-linear character of the non-elastic flow law, the non-elastic strain rate in response to the crack stress field is not a compatible strain rate field. Therefore, at each point in the body there is a time rate of accumulation of continuum dislocation density. This accumulation rate can be computed in terms of the spatial derivatives of $\dot{\epsilon}$. The accumulation rate results in a time rate of generation of self-stress in the body. At any time the total stress in the body is uniquely determined by the applied loads, the current dislocation density distribution and the boundary conditions imposed by the presence of the crack. Let us now examine a bit more closely how this comes about.

(a) *The dislocation density tensor*

Consider first the description of the state of strain of an elastic body that can also sustain plastic strains. Let u_i^T be the displacement field corresponding to the sum of elastic and plastic strains and let β_{ij}^T, β_{ij}^E and β_{ij}^P be respectively the total, elastic and plastic distortion tensors. Then by definition of these quantities

$$u_{i,j}^T = \beta_{ij}^T, \tag{9}$$

$$= \beta_{ij}^E + \beta_{ij}^P, \tag{10}$$

where

$$u_{i,j}^T \equiv \partial u_i^T / \partial x_j. \tag{11}$$

In general, the quantities β_{ij}^E and β_{ij}^P are not separately expressible as gradients of components of u_i^T. Because of eqn (9), the total distortion satisfies the condition that its integral about any closed path C bounding some surface S equals zero. Thus

$$\int_C \mathrm{d}x_j \beta_{ij}^T = 0 \tag{12}$$

then for the two terms in eqn (10)

$$0 = \int_C \mathrm{d}x_j \beta_{ij}^E + \int_C \mathrm{d}x_j \beta_{ij}^P, \tag{13}$$

$$= -\int_S \mathrm{d}s_j e_{jkl} \beta_{il,k}^E - \int_S \mathrm{d}s_j e_{jkl} \beta_{il,k}^P, \tag{14}$$

where we have employed Stokes theorem and e_{jkl} is the completely anti-symmetric 3-index symbol and $\mathrm{d}s_j$, is an oriented element of S. We define the dislocation density tensor, following Kröner[15], as

$$\alpha_{ij} = -e_{jkl} \beta_{il,k}^P. \tag{15}$$

Since eqn (14) is satisfied for arbitrary S, it follows then that

$$\alpha_{ij} = e_{jkl} \beta_{il,k}^E \tag{16}$$

and the dislocation density associated with the plastic field is reflected also by the elastic field and so a characteristic state of stress results.

There is an extensive literature on the relationship of the stress states resulting from such dislocation distribution and the associated incompatibility. Good reviews of the subject have been published by Mura[16] and deWit[17]. We shall not discuss here the general methods that are available but rather shall specialize our problem immediately to the two-dimensional anti-plane strain case.

For this case all displacements are in the direction normal to the plane. We shall take the coordinates in the plane to be x_1 and x_2 (or x and y) and the normal direction to be x_3 (or z). The only components of β_{ij} that remain are β_{3j} and $j = 1, 2$. Now the only non-zero component of α_{ij} is

$$\alpha_{33} = -e_{3kl}\beta^P_{3l,k} = -(\beta^P_{32,1} - \beta^P_{31,2}). \tag{17}$$

We shall write this in two-dimensional vector form and suppress the index 3. Thus

$$\alpha = -\nabla \times \boldsymbol{\beta}^P, \tag{18}$$

where

$$\boldsymbol{\beta}^P \equiv \beta_{31}\hat{\mathbf{x}}_1 + \beta_{32}\hat{\mathbf{x}}_2, \tag{19}$$

and $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ are unit basis vectors. Since there is no distinction in this geometry between the distortion $\boldsymbol{\beta}^P$ and the symmetric plastic strain $\boldsymbol{\epsilon}$ (other than a possible factor of 2) we shall replace $\boldsymbol{\beta}^P$ from here on by $\boldsymbol{\epsilon}$ and so we shall have

$$\alpha = -\nabla \times \boldsymbol{\epsilon}, \tag{20}$$

or for time rates

$$\dot{\alpha} = -\nabla \times \dot{\boldsymbol{\epsilon}}. \tag{21}$$

(b) *The associated self stress*

The stress at $\mathbf{r}$ in an infinite uniform (uncracked) body that is generated by $\alpha(\mathbf{r}')$ is readily computed from the stress function

$$\chi \equiv (G/2\pi) \ln R(\mathbf{r}, \mathbf{r}') \tag{22}$$

where

$$R \equiv |\mathbf{r} - \mathbf{r}'|. \tag{23}$$

The stress vector $\boldsymbol{\sigma}$ is defined as

$$\boldsymbol{\sigma} \equiv \sigma_{31}\hat{\mathbf{x}}_1 + \sigma_{32}\hat{\mathbf{x}}_2. \tag{24}$$

or for simplicity (and alternatively)

$$\boldsymbol{\sigma} = \sigma_1\hat{\mathbf{x}}_1 + \sigma_2\hat{x}_2, \tag{25}$$

$$= \sigma_x\hat{\mathbf{x}} + \sigma_y\hat{\mathbf{y}}. \tag{26}$$

For our case the result is

$$\sigma_x{}^\alpha(\mathbf{r}) = -(\partial/\partial y)(G/2\pi)\int \mathrm{d}s(\mathbf{r}')\alpha(\mathbf{r}') \ln R(\mathbf{r}, \mathbf{r}'), \tag{27}$$

$$\sigma_y{}^\alpha(\mathbf{r}) = (\partial/\partial x)(G/2\pi)\int \mathrm{d}s(\mathbf{r}')\alpha(\mathbf{r}') \ln R(\mathbf{r}, \mathbf{r}'), \tag{28}$$

where G is the modulus of rigidity and the integral is extended over the entire body.

If we consider the x, y-plane as the complex z-plane, where

$$z \equiv x + iy, \tag{29}$$

we can recast eqns (27) and (28) in a compact complex form as follows:

Let the complex stress function $\tilde{\sigma}^\alpha$ be given by

$$\tilde{\sigma}^\alpha \equiv \sigma_x{}^\alpha - i\sigma_y{}^\alpha. \tag{30}$$

Then eqns (27) and (28) are simply represented by

$$\tilde{\sigma}^\alpha = \frac{G}{2\pi i}\int \mathrm{d}s'\alpha(z')\frac{1}{z-z'}. \tag{31}$$

As noted above, the kernel of the integral of eqn (31) is that for the uncracked body. For the problem to be pursued below it is important to have the kernel for the semi-infinite crack in the infinite plane. The geometry of the problem is shown in Fig. 1. The boundary conditions required for the stress due to the dislocation density α is that for $y=0$ and $x<0$, the stress component $\sigma_y{}^\alpha$ must vanish. Thus on the crack faces $\tilde{\sigma}$ must be purely real. The resulting formula for $\tilde{\sigma}^\alpha$ for the case that

$$\alpha(\bar{z}) = \alpha(z), \tag{32}$$

i.e. for α symmetrical with respect to the x-axis, is

$$\tilde{\sigma}^\alpha(z) = \frac{G}{2\pi i}\int \mathrm{d}s'\alpha(z')\sqrt{\left(\frac{z'}{z}\right)}\frac{1}{z-z'}. \tag{33}$$

The integral in eqn (33) extends over the entire plane. This result is derived in Appendix A below.

The symmetry condition stated by eqn (32) is satisfied by all the configurations with which we are concerned in the theory.

4. DEVELOPMENT OF THE THEORY

We are now ready to proceed with the exposition of the theory. The configuration that will be investigated is one for which a precise translational steady state can be specified, viz. a semi-infinite crack, extending from $-\infty$ to 0 on the x-axis, in an otherwise infinite body. The deformations and loadings are those of anti-plane strain and anti-plane stress simultaneously. The configuration is illustrated in Fig. 1. The crack tip, which is at the origin of the coordinates, is considered to be moving to the right at a velocity v and so the crack is extending at that velocity.

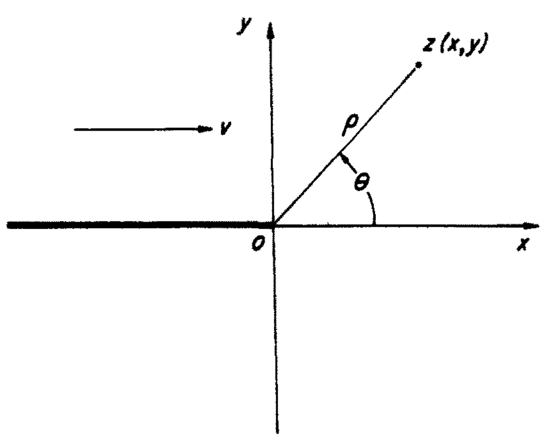

Fig. 1. The crack geometry for the semi-infinite Mode III crack. The crack occupies the negative x-axis. The crack tip is moving to the right with velocity v.

The problem is characterized by two constitutive laws, as discussed above. These are:

(a) *Crack kinetic law*

$$v = v(\mathcal{G}, \mathcal{G}_0, T), \tag{3}$$

$$\mathcal{G} \equiv K^2/2G. \tag{34}$$

(b) *Non-elastic flow law*

$$\dot{\boldsymbol{\epsilon}} = (\dot{\epsilon}/\sigma)\boldsymbol{\sigma}, \tag{4}$$

$$\dot{\epsilon} = \dot{\epsilon}^*(\sigma^*)\phi(\sigma/\sigma^*). \tag{8}$$

In addition the body exhibits elastic strain proportional to the stress $\boldsymbol{\sigma}$ according to Hooke's Law.

The remote loading of the body is stated as a condition on the asymptotic stress field in the body. This stress field $\boldsymbol{\sigma}^A$ is to be considered as the applied stress on the body. It has the form

$$\boldsymbol{\sigma}^A = \frac{K_A}{\sqrt{(2\pi)}} \frac{1}{\sqrt{\rho}} \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}, \tag{35}$$

where ρ and θ are polar coordinates of the field point, K_A is the applied (or apparent) stress intensity factor characteristic of the remote loading and the last factor of the r.h.s. is a column vector symbol in which the upper entry is the x-component. If the body were solely linear elastic, $\boldsymbol{\sigma}^A$ would be the complete stress field for the cracked body.

For the general case under consideration, the effect of prior deformation is entirely characterized by the current dislocation density $\alpha(\boldsymbol{r})$. Because of the linearity of stress fields we can represent the total stress $\boldsymbol{\sigma}$ as the sum of $\boldsymbol{\sigma}^A$ and $\boldsymbol{\sigma}^\alpha$, the stress produced by α. Thus

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^A + \boldsymbol{\sigma}^\alpha. \tag{36}$$

In complex form

$$\tilde{\sigma}^A = \frac{1}{i} \frac{K_A}{\sqrt{(2\pi)}} \frac{1}{z^{1/2}}, \tag{37}$$

and, as shown above,

$$\tilde{\sigma}^\alpha = \frac{G}{2\pi i} \int \mathrm{d}s(z')\alpha(z') \sqrt{\left(\frac{z'}{z}\right)} \frac{1}{z - z'}. \tag{33}$$

Then

$$\tilde{\sigma} = \tilde{\sigma}^A + \tilde{\sigma}^\alpha. \tag{38}$$

In order for this representation to be meaningful, it is necessary only that the integral in eqn (33) exist. We shall show below that the integral does exist for a broad range of circumstances.

The stress given by eqn (33) can be analyzed more closely as follows:

By simple algebraic manipulation, $\tilde{\sigma}^\alpha$ can be written in the form

$$\tilde{\sigma}^\alpha = -\frac{G}{2\pi i} \frac{1}{\sqrt{z}} \int \mathrm{d}s'\alpha(z') \frac{1}{\sqrt{z'}} + \frac{G}{2\pi i} \sqrt{z} \int \mathrm{d}s'\alpha(z') \frac{1}{\sqrt{z'}} \frac{1}{z - z'}. \tag{39}$$

Now, the second term of the r.h.s. is non-singular, while the first term is singular. It is the singular term that is of interest to us. If we define

$$K_p \equiv -\frac{G}{\sqrt{(2\pi)}} \int \mathrm{d}s' \frac{\alpha(z')}{\sqrt{z'}}, \tag{40}$$

then the singular part of $\tilde{\sigma}^{\alpha}$ is of the form

$$\tilde{\sigma}^{\alpha}_{\text{sing}} = \frac{1}{i}\frac{K_p}{\sqrt{(2\pi)}}\frac{1}{z^{1/2}} \tag{41}$$

which, of course, strongly resembles $\tilde{\sigma}^A$. Then we can express the singular part of $\tilde{\sigma}$ as

$$\tilde{\sigma}_{\text{sing}} = \frac{1}{i}\frac{K}{\sqrt{(2\pi)}}\frac{1}{z^{1/2}}, \tag{42}$$

where

$$K = K_A + K_P. \tag{43}$$

This analysis of the singular behavior of $\tilde{\sigma}$ leads to the important result that we can consider the K that characterizes the actual crack tip field as the simple sum of the "applied" part K_A and a "plastic" part K_P. Of course, K_P is generally negative. We can then characterize all the effect of non-elastic flow by the value of K_P.

Now we can state our problem precisely as follows.

(1) Given a fixed crack tip velocity v and a fixed value of K_A, find the value of K_P (or of K) that corresponds to a steady state for all the field variables, or stated as an equation,

$$K_P = K - K_A = K_P(K_A, v). \tag{44}$$

(2) Combine this result with the constitutive law given in as eqn (3) and restated as

$$v = g(K). \tag{45}$$

By elimination of K_P between the two equations, the desired result will be obtained as

$$v = f(K_A). \tag{46}$$

We now proceed to the first sub-problem, which is simply a problem in solid mechanics.

Now it is sufficient, to specify the steady state, to prescribe that, in a reference frame moving with a velocity v, the value of $\alpha(r)$ is stationary with respect to time. This is the same as the requirement

$$0 = \dot{\alpha} + v\partial_x\alpha, \tag{47}$$

where by $\dot{\alpha}$ is meant the time rate of change of α at a fixed point in the fixed frame of reference.

Then, subject to the condition of eqn (47) we wish to compute

$$K_p = -\frac{G}{\sqrt{(2\pi)}} \int \mathrm{d}s \frac{\alpha(z)}{\sqrt{z}}. \tag{40}$$

The value of K_p seems to depend on all prior deformation history and on the details of α. However, by the following trick, we can transform the integral into one that depends only on the current deformation rate in the vicinity of the crack tip.

Note that, for steady state α,

$$\int \mathrm{d}s z^{1/2}\dot{\alpha} = -\mathbf{v}\cdot\int \mathrm{d}s z^{1/2}\nabla\alpha, \tag{48}$$

$$= \mathbf{v}\cdot\int \mathrm{d}s\alpha\nabla z^{1/2} - \mathbf{v}\cdot\int \mathrm{d}r\hat{n}z^{1/2}\alpha, \tag{49}$$

where the last integral of the r.h.s. of eqn (49) extends over a contour at infinity bounding the body, dr is a differential linear element of that contour and $\hat{n}$ is the unit, outward direction normal, at each point of the contour. The bounding contour integral goes to zero as the boundary tends to infinity and so

$$\int \mathrm{d}s z^{1/2}\dot{\alpha} = v\int \mathrm{d}s\alpha\partial_x z^{1/2}, \tag{50}$$

$$= (v/2)\int \mathrm{d}s\frac{\alpha}{\sqrt{z}}. \tag{51}$$

Then, from eqn (40),

$$K_p = -\frac{2}{v}\frac{G}{\sqrt{(2\pi)}}\int \mathrm{d}s z^{1/2}\dot{\alpha}, \tag{52}$$

$$= \frac{2}{v}\frac{G}{\sqrt{(2\pi)}}\int \mathrm{d}s z^{1/2}\nabla\times\dot{\boldsymbol{\epsilon}}. \tag{53}$$

The calculation to this point is exact and it emphasizes the dependence of K_p on the rate of deformation in the vicinity of the crack tip. In order to obtain a solution of the problem, however, we must now introduce an approximation. We shall approximate the stress field $\boldsymbol{\sigma}$, that generates $\dot{\boldsymbol{\epsilon}}$, by the singular part of $\boldsymbol{\sigma}$ given in eqn (42) above. This approximates $\boldsymbol{\sigma}$ well in those regions where the stress is largest and where $\dot{\alpha}$ is largest. This reflects the strong dominance of the crack face boundary conditions in the vicinity of the crack tip as appears in eqn (39).

We write therefore, as an approximation for $\boldsymbol{\sigma}$,

$$\boldsymbol{\sigma} = \frac{K}{\sqrt{(2\pi)}}\frac{1}{\sqrt{\rho}}\begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}, \tag{54}$$

$$\sigma = \frac{K}{\sqrt{(2\pi)}}\frac{1}{\sqrt{\rho}}, \tag{55}$$

and, if $\hat{\boldsymbol{\sigma}}$ denotes a unit vector with the direction of $\boldsymbol{\sigma}$,

$$\hat{\boldsymbol{\sigma}} = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}. \tag{56}$$

With this choice for $\boldsymbol{\sigma}$,

$$\nabla\times\dot{\boldsymbol{\epsilon}} = -(\dot{\epsilon}/\sigma)[(n-1)\sigma\nabla\times\hat{\boldsymbol{\sigma}} - n\nabla\times\boldsymbol{\sigma}], \tag{57}$$

$$\nabla\times\hat{\boldsymbol{\sigma}} = \frac{1}{2}\frac{1}{\rho}\cos\frac{\theta}{2}, \tag{58}$$

$$\nabla\times\boldsymbol{\sigma} = 0, \tag{59}$$

where n is given by

$$n \equiv (\partial \ln \dot{\epsilon}/\partial \ln \sigma)_{\sigma^*}. \tag{60}$$

Making the appropriate substitutions in eqn (53), we obtain

$$K_p = -\frac{2}{v}\frac{G}{\sqrt{(2\pi)}}\int \mathrm{d}sz^{1/2}\cdot\frac{1}{2}\frac{1}{\rho}\cos\frac{\theta}{2}\cdot(n-1)\dot{\epsilon}, \tag{61}$$

$$= -\frac{1}{v\sqrt{(2\pi)}}\frac{G}{}\int_{-\pi}^{\pi}\mathrm{d}\theta\cos^2\frac{\theta}{2}\int_0^{\infty}\mathrm{d}\rho\sqrt{\rho}(n-1)\dot{\epsilon}, \tag{62}$$

$$= -\frac{1}{2}\frac{1}{v}G\sqrt{(2\pi)}\int_0^{\infty}\mathrm{d}\rho\sqrt{\rho}(n-1)\dot{\epsilon}. \tag{63}$$

Now, from eqn (55),

$$\sqrt{\rho}\,\mathrm{d}\rho = \frac{1}{(2\pi)^{3/2}}\left(\frac{K}{\sigma^*}\right)^3\frac{\sigma^*}{\sigma}\mathrm{d}\left(\frac{\sigma^*}{\sigma}\right)^2, \tag{64}$$

$$= -\frac{2}{(2\pi)^{3/2}}\left(\frac{K}{\sigma^*}\right)^3\frac{\mathrm{d}\eta}{\eta^4}, \tag{65}$$

where

$$\eta \equiv \sigma/\sigma^*. \tag{66}$$

This leads to a convenient modification of eqn (63) as

$$K_P = -\frac{1}{2\pi}\frac{\dot{\epsilon}^*}{v}\frac{K^3G}{\sigma^{*3}}\int_0^{\infty}\mathrm{d}\eta\frac{(n-1)\phi(\eta)}{\eta^4}, \tag{67}$$

and, of course, n depends only on η.

We have obtained our principal result without committment to a special form for $\phi(\eta)$. The only requirement on ϕ is that the integral in eqn (67) exists. For ϕ as a monotonic positive function of η, that requirement is simply that: (a) as $\eta \to 0$, ϕ approaches zero faster than η^3 and (b) as $\eta \to \infty$, ϕ becomes unbounded more slowly than η^3. This requirement is easily met by real constitutive laws for non-elastic deformation.

Note, incidentally, that if $\phi(\eta)$ is linear in η (Newtonian flow), then $n = 1$ and $K_P = 0$. Thus for a body that undergoes Newtonian non-elastic deformation, $K = K_A$ and there is no distinction from the linear elastic case.

In any case, the integral is dimensionless and is independent of the loading. We shall reserve discussion of the effect of particular constitutive laws until the next section.

We shall now rewrite eqn (67) in a simpler form as

$$K_p = -AP\frac{K^3}{v}, \tag{68}$$

where we define

$$A \equiv \frac{1}{2\pi}\frac{\dot{\epsilon}^*G}{\sigma^{*3}}, \tag{69}$$

and

$$P \equiv \int_0^{\infty}\mathrm{d}\eta\frac{(n-1)\phi(\eta)}{\eta^4} \tag{70}$$

Now, from eqn (43),

$$K_p = K - K_A, \tag{71}$$

and so

$$K_A = K + AP\frac{K^3}{v}. \tag{72}$$

This equation represents our principal result for the steady state and it remains only to introduce v as a function of K in order to complete the second step of our problem plan stated above.

For the purpose of this paper we shall choose a form for v that is suggested by the work of Lawn[12]. We have modified Lawn's result only by replacing an exponential by a hyperbolic sine. This insures that the crack tip velocity is zero when the Griffith–Irwin equilibrium condition is satisfied. We write then

$$v = v^* \sinh\frac{K^2 - K_0^2}{K_1^2} \tag{73}$$

where K_0 is the value of K corresponding to $\mathcal{G}_0$, K_1 is a quantity that depends only on temperature and material constants and v^* is a rate factor that depends on temperature through an Arrhenius factor.

We cannot eliminate K between eqns (72) and (73) in simple closed form and so we shall show the dependence of v upon K_A numerically, in the next section, for a typical case. It is in fact rather more illuminating to consider first the dependence of K upon K_A and we shall proceed immediately to that demonstration.

5. THE CHARACTER OF THE STEADY STATE

From eqns (72) and (73) we obtain easily the equation

$$K_A = K + \frac{AP}{v^*}\frac{K^3}{\sinh[(K^2 - K_0^2)/K_1^2]}. \tag{74}$$

This equation contains many material constants the values of which can cover a wide range, depending on the particular material, temperature and environmental conditions and the reliability of our representation of $\dot{\epsilon}$ and v. The importance of our current discussion is, however, the general form of the solution and so we shall reduce the dependence of our equation on material constants to the simplest combinations possible. We therefore introduce reduced dimensionless measures for K as follows:

Let

$$k \equiv K/K_1, \tag{75}$$

$$k_0 \equiv K_0/K_1, \tag{76}$$

$$k_A \equiv K_A/K_1. \tag{77}$$

Now eqn (74) becomes

$$k_A = k + B\frac{k^3}{\sinh(k^2 - k_0^2)}, \tag{78}$$

where

$$B \equiv APK_1^2/v^*. \tag{79}$$

Thus we have an equation whose character depends only on the numbers B and k_0. We can now explore the character of eqn (78) for a significant range of values for those constants.

In Fig. 2 we present plots of k_A vs k, according to eqn (78), for the parameter values $k_0 = 0.1$ and $B = 1$ and 5. With these two values of B we exhibit two types of curve that can result. That for $B = 5$ has three extremals, two minima and one maximum. The one for $B = 1$ has only one minimum. At some value for B, slightly higher than 1, there is a minimum and a point of zero curvature at which a maximum and a minimum coincide. The portions of each curve for which k_A decreases with increasing k are distinguished by dashed lines. Those portions of the steady state curves represent unstable branches. That such branches are unstable regimes follows from the circumstance that on those branches k increases as k_A decreases and, since v is monotonic in k, the velocity is increasing with decreasing "applied" k.

The more interesting curve is that for $B = 5$. That case exhibits two stable branches that share overlapping ranges of k_A. In Fig. 3 we show plots of log v (in arbitrary units) vs k_A for both stable branches. This is the plot that is to be compared with experimental curves of log v vs apparent k. Note that what we have designated by k_A is the k that is customary for characterizing experimental conditions.

A prominent feature of both branches in Fig. 3 is the vertical slope of the curves at the limits of the stable regimes. This feature follows directly from the horizontal slopes of k_A vs k at the stability limits in that plot. *Note especially that it signals no special micromechanism.* It is indeed striking that this ubiquitous feature of experimental log v vs k_A curves follows unequivocally from the theory presented here. Thus the theory predicts threshhold values for k_A as well as "critical" values at which the crack appears to "runaway". The lower threshhold value of k_A, in our example of Fig. 3 is practically fifteen times the equilibrium k_0 on which the plot is based. We shall not explore further numerical cases in this paper.

The region of k_A between k_0 and the lower threshhold is also interesting. In that range of k_A non-steady state crack extension is possible. That regime of crack extension can result in considerable crack growth under cyclic loading. This is, of course, fatigue behavior. In order to discuss that we must consider the non-steady state behavior. We shall do that in the next section.

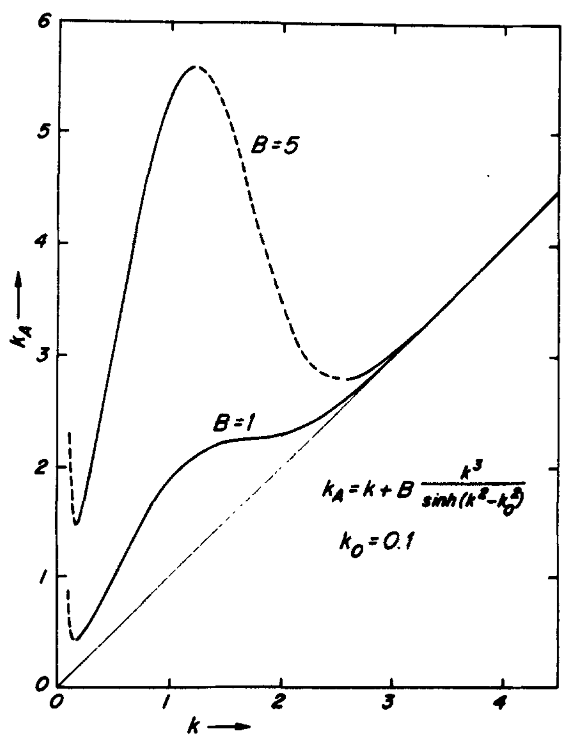

Fig. 2. Plots of "apparent" stress intensity factor k_A vs actual stress intensity factor k according to eqn (78). The equilibrium stress intensity factor k_0 has the value 0.1. Curves are shown for $B = 5$ with bi-modal behavior and for $B = 1$ with a single mode. The diagonal line is the locus $k_A = k$ that would correspond to the purely elastic case.

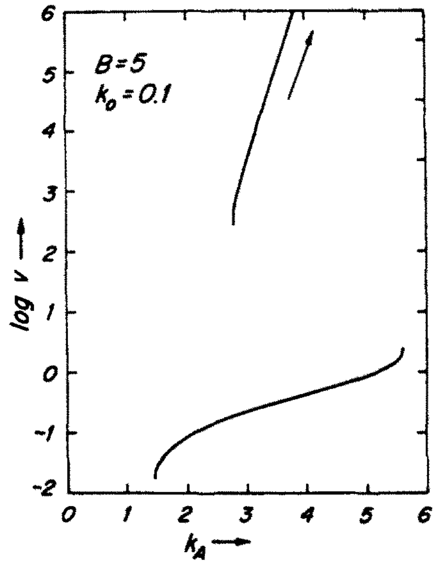

Fig. 3. Plots of log v vs k_A for the case $B = 5$ from Fig. 2. The lower branch is an observable steady state velocity regime. The upper branch exhibits so strong a dependence of velocity on k_A, that it would probably be characterized as "run-away" growth in normal experimentation.

6. THE NON-STEADY STATE

We shall now seek an answer to the question: Given that the current values of K_A and K_P (or of K_A and K) do not correspond to the steady state condition, how does this state relax to the steady state?

Unfortunately, there is not a unique answer to the question with the conditions given. The reason for this is that the values of K_A and K_P do not uniquely characterize the non-steady state of the cracked body. The same values of K_A and K_P may be reached through more than one loading history and so different subsequent histories of K_P may result when K_A is held constant.

It may be expected, nevertheless, that, for a limited range of loading histories, the variations in relaxation among the several states characterized by the same K_A and K_P may be slight and that there is one principal relaxation path that accounts for most of the resultant relaxation history. By placing suitable restrictions on the general state of crack, that are consistent with the steady state solutions, we can develop such a principal dependence of the time rate of change of K_P upon the current values of K_A and K_P as follows:

If α is the current distribution of dislocation density, then in a coordinate frame moving in the x-direction with the crack tip velocity v, the time rate of change of α is

$$\frac{d\alpha}{dt} = \dot{\alpha} + v\partial_x\alpha, \tag{80}$$

where, as before, $\dot{\alpha}$ is the time rate of change of α at a point fixed in the material frame.

Since

$$K_P = -\frac{G}{\sqrt{(2\pi)}}\int ds\,\frac{\alpha(z)}{\sqrt{z}}, \tag{40}$$

we can define

$$\frac{dK_P}{dt} \equiv -\frac{G}{\sqrt{(2\pi)}}\int ds z^{-1/2}\frac{d\alpha}{dt}, \tag{81}$$

$$= -\frac{G}{\sqrt{(2\pi)}}\int ds z^{-1/2}[\dot{\alpha} + v\partial_x\alpha]. \tag{82}$$

Define further the quantities

$$\frac{\partial K_P}{\partial t} \equiv -\frac{G}{\sqrt{(2\pi)}} \int \mathrm{d}s z^{-1/2} \dot{\alpha}, \tag{83}$$

and

$$\frac{\partial K_P}{\partial a} \equiv -\frac{G}{\sqrt{(2\pi)}} \int \mathrm{d}s z^{-1/2} \partial_x \alpha, \tag{84}$$

where $\partial K_P/\partial t$ is the time rate of change of K_P due only to the current deformation rate and $\partial K_P/\partial a$ is the change of K_P caused by unit displacement of the crack tip. We can now rewrite eqn (82) in the suggestive form

$$\frac{\mathrm{d}K_P}{\mathrm{d}t} = \frac{\partial K_P}{\partial t} + v \frac{\partial K_P}{\partial a}. \tag{85}$$

Now, with the same singular stress approximation for $\boldsymbol{\sigma}$ that we employed to evaluate K_P (beginning at eqn 52), we can evaluate $\partial K_P/\partial t$. We omit the detailed calculation, which is completely analogous to that leading to eqn (67) and report the result to be

$$\frac{\partial K_P}{\partial t} = -\frac{1}{2} \dot{\epsilon}^* G(K/\sigma^*) Q, \tag{86}$$

where

$$Q \equiv \int_0^\infty \frac{\mathrm{d}\eta}{\eta^2} (n-1)\phi(\eta). \tag{87}$$

We cannot evaluate $\partial K_P/\partial a$ directly. However, that quantity will be completely consistent with the requirements at each steady state point if we assign it the value

$$\frac{\partial K_P}{\partial a} = -\pi(\sigma^*/K)^2 K_P (Q/P), \tag{88}$$

where, as above,

$$P \equiv \int_0^\infty \frac{\mathrm{d}\eta}{\eta^4} (n-1)\phi(\eta). \tag{70}$$

Now, if we write $\dot{K}_P$ for $\mathrm{d}K_P/\mathrm{d}t$, we obtain the answer

$$\dot{K}_P = -\frac{1}{2} Q \dot{\epsilon}^* G(K/\sigma^*) - \pi(Q/P) v (\sigma^*/K)^2 K_P. \tag{89}$$

when steady state is reached, $\dot{K}_P = 0$ and

$$K_P = -\frac{G}{2\pi} \frac{s\dot{\epsilon}^*}{v} \left(\frac{K}{\sigma^*}\right)^3 P. \tag{90}$$

The assignment made at eqn (88) is of course nothing more than evaluating $\partial K_P/\partial a$ at a steady state point and then extending that expression beyond the steady state curve. The important point of this is that all steady state points with the same value of K_P are mutually consistent according to our prescription.

We note that, for any fixed choice of material parameters, $\dot{K}_P$ is negative for the region of the (k, k_A)-plane below the steady state curve and is positive for the region above that curve.

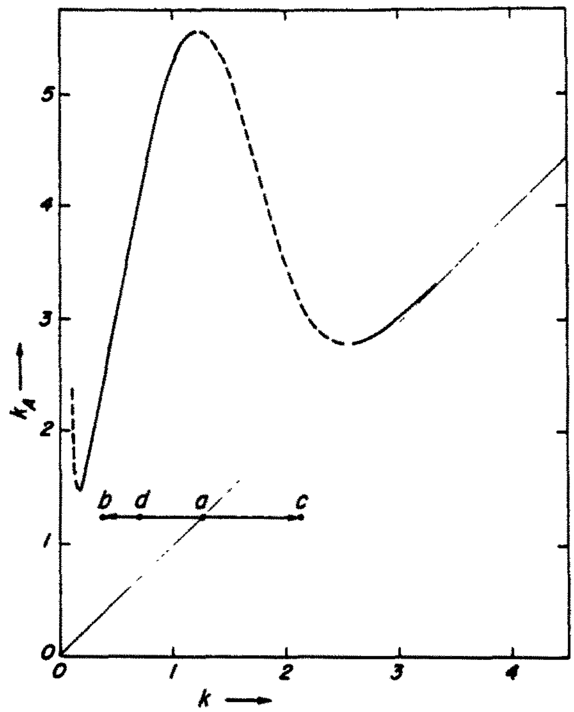

Fig. 4. A replot of the curve for $B = 5$ from Fig. 2. Loading and response path is shown for cyclic loading at levels of k_A below the steady state threshhold. This fatigue cracking behavior is described in Section 7.

The theory presented here predicts a strong effect on crack extension from cycling the value of k_A. Note in our discussion here that crack propagation occurs with either positive or negative loading because of the symmetrical nature of shear cracking.

Consider, then, the application to the cracked body of alternate values of k_A of the same magnitude but of alternating sign. This magnitude of k_A is that which would be termed the loading amplitude Δk in a cyclic loading experiment. We consider the individual loadings to be held constant for a time interval τ.

The initial loading occurs when the value of k_P is zero, corresponding to the point a on Fig. 4. During the first loading period the value of k_P will become negative according to eqn (89) until at the time τ it reaches a value for which k corresponds to the point b. During that time the crack will extend by an amount Δa given by

$$\Delta a = \int_0^\tau \mathrm{d}t v(k). \tag{91}$$

The value of v at each time depends only on the value of k that is then current and that is determined by eqn (89).

If at that time k_A is reversed and, if we continue to use the positive k_A axis to represent the further behavior, then relative to k_A we may consider k_P to become reversed. Thus k_P, which had become strongly negative, now appears to be positive with the same magnitude and *assists further crack extension*. We represent this state upon reversal by the point c on Fig. 4. Now, during the next time interval τ, that system state point moves to the left until it reaches some different end point d. In that time interval considerable further crack extension has occurred. This process clearly can continue and ultimately reach a relatively stable amplitude for k_P and a fairly constant value for Δa at each cycle.

Of course cyclic loading can also be carried out at higher levels of k_A for which a considerable portion of each cycle interval can even be spent at or near the appropriate steady state point.

Clearly it is possible to describe fatigue behavior with the present theory without introducing any new rules.

8. CONCLUSIONS

We have presented a theory for crack propagation in ductile materials that is based on simple realistic assumptions and that describes most of the observed phenomena of crack propagation. An important feature of the theory is that it predicts crack behavior continuously as it depends on loading history. The principal distinction from prior theory is that a crack velocity kinetic law replaces the customary static fracture criteria.

It has been shown that, for a moving crack tip and for time dependent deformation, the Irwin crack extension force is defined. The crack extension process is then describable by a kinetic extension of the equilibrium condition as in the "brittle" case. The crack extension force is dependent on the stress intensity factor K that characterizes the crack tip singularity. That value K was shown to be the sum of the apparent K_A, determined from the remote loading and a "plastic" contribution K_P. Thus

$$K = K_A + K_P. \tag{43}$$

The principal result of the calculation is that, at steady state,

$$K_P = -APK^3/v, \tag{68}$$

where v is the crack tip velocity, A depends on material modulus and flow parameters and P is a dimensionless integral that is some moment of the constitutive function. An expression was also derived (eqn 89) for the principal relaxation mode from a non-steady state value of K_P to the steady state.

With these results and with a reasonable expression for v as a function of K, the dependence of steady state velocity on the value of K_A was deduced. The curve of v vs K_A exhibited both threshhold and "critical" behavior due to stability limits predicted by the theory. Below the steady state regime a time dependent behavior under cyclic loading was demonstrated, that could be identified with fatigue crack growth.

Acknowledgements—The research reported here has been supported by the National Science Foundation through the Mechanical Studies Program of the Cornell University Materials Science Center.

REFERENCES

1. A. A. Griffith, The phenomena of rupture and flow in solids. *Philosophical Transactions of the Royal Society* (*London*) **A221**, 163 (1921).
2. G. R. Irwin, Analysis of stresses and strains near the end of a crack traversing a plate. *Trans. ASME, J. Appl. Mech.* **24**, 361 (1957).
3. G. I. Barenblatt, The mathematical theory of equilibrium cracks in brittle fracture. *Advances in Applied Mechanics* Vol. 7, p. 55. Academic Press (1962).
4. J. W. Hutchinson, Singular behavior at the end of a tensile crack in a hardening material. *J. Mech. and Phys. of Solids* **16**, 13 (1968); Plastic stress and strain fields at a crack tip, *Ibid.* **16**, 337 (1968).
5. J. R. Rice and G. F. Rosengren, Plane strain deformation near a crack tip in a power-law hardening material. *J. Mech. Phys. of Solids* **16**, 1 (1968).
6. R. E. Peierls, *Proc. Phys. Soc.* **52**, 23 (1940).
7. R. Thomson, C. Hsieh and V. Rana, Lattice trapping of fracture cracks. *J. Appl. Phys.* **42**, 3154 (1971).
8. G. I. Barenblatt, V. M. Entov and R. L. Sagalnik, *Inelastic behavior of solids* (Edited by M. F. Kanninen *et al.*), p. 559. McGraw-Hill, New York (1970).
9. P. C. Gehlen and M. F. Kanninen, *Inelastic behavior of solids* (Edited by M. F. Kanninen *et al.*), p. 587, McGraw-Hill, New York (1970).
10. C. Hsieh and R. Thomson, Lattice theory of fracture and crack creep. *J. Appl. Phys.* **44**, 205 (1973).
11. B. R. Lawn and T. R. Wilshaw, *Fracture of brittle solids.* Cambridge University Press, Cambridge (1975).
12. B. R. Lawn, An atomistic model of kinetic crack growth in brittle solids. *J. Mat. Sci.* **10**, 469 (1975).
13. E. W. Hart, Constitutive relations for the non-elastic deformation of metals. *Trans. ASME, J. Eng. Matls. and Tech.* **98**, 193 (1976).
14. T. Mura, Continuum theory of plasticity and dislocations. *Int. J. Engng Sci.* **5**, 341 (1967).
15. E. Kröner, Kontinuums theorie der versetzungen und eigenspannungen. *Ergeb. Angew. Math.* Vol. 5. Springer, Berlin (1958).
16. T. Mura, *Advances in materials research* (Edited by H. Herman), Vol. 3, p. 1. Interscience Publishers, New York (1968).
17. R. deWit, The continuum theory of stationary dislocations. *Solid State Physics* **10**, 249 (1960).
18. B. A. Bibly and J. D. Eshelby, *Fracture* (Edited by H. Liebowitz), Vol. 1, p. 121. Academic Press, New York (1968).

APPENDIX A

The formula given in eqn (33) is derived from eqn (31) by inclusion of the crack boundary conditions in the following way. From eqn (31) we compute the value of σ_y^α on the left semi-infinite interval of the x-axis. After the crack is introduced on that same interval, the stress field is left undisturbed if equal and opposite tractions are applied to the two crack faces of magnitude equal to σ_y^α. The relaxation of those tractions to zero results in the introduction of a complementary field $\tilde{\sigma}_c^\alpha$. The resultant stress field of the cracked body is then the sum of the stress field $\tilde{\sigma}_\infty^\alpha$ (eqn 31) for the uncracked body and of the complementary stress $\tilde{\sigma}_c^\alpha$. Our procedure is an extension of that described by Bilby and Eshelby [18] for the finite crack.

From eqn (31), with $\alpha(z)$ satisfying the symmetry condition of eqn (32),

$$\sigma_y^\alpha(x,0) = \frac{G}{2\pi}\int ds'\alpha(z')\frac{1}{x-z'}. \tag{A.1}$$

The complementary field is given by the well-known Cauchy integral

$$\tilde{\sigma}_c^\alpha = -\frac{1}{\pi}\int_{-\infty}^{0} dx\sigma_y^\alpha(x,0)\sqrt{\left(\frac{x}{z}\right)}\frac{1}{z-x}. \tag{A.2}$$

Substitution of eqn (A.1) in eqn (A.2) yields

$$\tilde{\sigma}_c^\alpha = -\frac{G}{\pi}\int ds'\alpha(z')\frac{1}{2\pi}\int_{-\infty}^{0} dx\sqrt{\left(\frac{x}{z}\right)}\frac{1}{(z-x)(x-z')}. \tag{A.3}$$

The integral over x is readily transformed into one over the integration variable z'' and over the path C that runs from $-\infty$ to zero along the upper face of a cut that coincides with the crack, circling the origin in a clockwise sense, returning along the lower face of the cut to $-\infty$ and then being completed by a circle at infinity, traversed counter-clockwise to the starting point. The resulting integral is

$$\tilde{\sigma}_c^\alpha = \frac{G}{2\pi i}\int ds'\alpha(z')\cdot\frac{1}{2\pi i}\int_C dz''\sqrt{\left(\frac{z''}{z}\right)}\frac{1}{(z-z'')(z''-z')}, \tag{A.4}$$

where we have introduced imaginary factors for convenience and the extra factor of one-half results from the double traverse of the negative x-axis. Upon evaluation from the residues at the poles at z and z',

$$\tilde{\sigma}_c^\alpha = -\frac{G}{2\pi i}\int ds'\alpha(z')\frac{1}{z-z'}+\frac{G}{2\pi i}\int ds'\alpha(z')\sqrt{\left(\frac{z'}{z}\right)}\frac{1}{z-z'}. \tag{A.5}$$

Finally, the stress for the cracked body is

$$\tilde{\sigma}^\alpha = \tilde{\sigma}_\infty^\alpha + \tilde{\sigma}_c^\alpha \tag{A.6}$$

$$= \frac{G}{2\pi i}\int ds'\alpha(z')\sqrt{\left(\frac{z'}{z}\right)}\frac{1}{z-z'}, \tag{A.7}$$

which is the formula given above as eqn (33).